

Hierarchical Wave Functions Revisited

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Abstract

We study the hierarchical wave functions on a sphere and on a torus. We simplify some wave functions on a sphere or a torus using the analytic properties of wave functions. The open question, the construction of the wave function for quasielectron excitations on a torus, is also solved in this paper.

ICTP HE 97

PACS: 73.40.Hm, 73.20.Dx, 03.65.-w, 03.80.+r

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1 Introduction

The trial wave function of the fractional quantum Hall effect (FQHE) on a disk at filling $\nu = 1/m$ with m an odd integer is given by the famous Laughlin wave function [1]. Laughlin wave function had been constructed on a sphere or a toric surface later in refs. [2, 3]. When $\nu \neq 1/m$, there are several proposals for constructing the trial wave function, notably Halperin's hierarchical wave function [4, 2] and Jain's composite fermion (CF) wave

function [5]. In this paper, we will only discuss the construction of the hierarchical wave function (for a review of FQHE, see ref. [6]).

Halperin's theory for the FQHE at a hierarchical filling is based on the picture of hierarchical condensations of quasiparticles. The charge and statistics of quasiparticles are both fractional, and those quasiparticles were recently observed in experiments [7].

For example, there are quasiparticle excitations in the case of Laughlin's state ($\nu = 1/m$). If those quasiparticles condense and form a Laughlin state them-self, a new fractional quantum Hall (FQH) state appears with $\nu = 1/(m - p)$ or $\nu = 1/(m + p)$ (p is an even integer), where the minus sign is due to the condensation of quasielectrons (QEs), and the plus sign is due to the condensation of quasiholes (QHs). The condensation of quasiparticles at $\nu = 1/(m - p)$ or $\nu = 1/(m + p)$ will lead to a higher hierarchical state. The process can continue and will form more complicated states.

Halperin also constructed the wave function at the hierarchical filling. The hierarchical wave function had been also further investigated in refs. [8, 9]. The hierarchical wave function is simply equal to the multiplication of those Laughlin wave functions in all hierarchical levels with all quasiparticle coordinates integrated out (it will become clear in the next section).

To construct the hierarchical wave function, we need to construct the Laughlin wave function with quasiparticle excitations. The Laughlin wave function in the presence of QH excitations is easy to construct, so does the hierarchical wave function due to the condensation of QHs (however the constructed wave function can not be analytically integrated). On the other hand, the Laughlin wave function in the presence of QE excitations is quite complicated and involves derivative operators in the case of a plane or a sphere. Thus the hierarchical wave function due to QE condensations is quite complicated and intractable in its old form. We also note that, the Laughlin wave function with QE excitations on a torus is still unknown, and thus the hierarchical wave function on a torus due to the condensation of QEs has not been obtained yet. We comment that, although refs. [10, 11] had constructed a hierarchical wave function at a hierarchical filling on a torus due to the condensation of QEs (such wave function on a sphere was first proposed in ref. [8]), the constructed wave function contains poles (or singularities) which are difficult to control.

Recently, some progresses in constructing the hierarchical wave function had been made in ref. [12]. In ref. [12], it was discovered that the Laughlin

wave function in the presence of QEs can be obtained by projecting a rather simple wave function to the lowest Landau level. Using this fact and the analytic properties of the wave functions of quasiparticles at all hierarchical levels, the hierarchical wave function due to the condensations of QEs can be greatly simplified (the wave function can be integrated analytically). We note that most of observed states in experiments are due to the condensation of QEs, only very few observed states involves the condensation of QHs.

In this paper, we will generalize the construction to the case when the surface is a torus. How to construct a Laughlin wave function with QE excitations on a torus is an open problem as noted in ref. [3]. We will solve this problem by proposing explicitly a Laughlin wave function with QE excitations on a torus which satisfies all required properties. Based on this construction, a hierarchical wave function due to the condensation of QEs on a torus can be constructed. No singularity exists in the wave function, and the wave function is analytically computable.

In ref. [13], the hierarchical wave function due to condensations of QHs was constructed. The wave function of QHs on a torus is multi-component and the multi-component nature of the QH wave function manifests clearly in the hierarchical wave function of electrons on a torus.

In refs. [10, 11], a hierarchical wave function on a torus due to condensations of QEs is constructed, although this particular wave function contains poles. In the contrary to the case of QHs [13], the multi-component nature of the wave function of QEs does not show in the wave function. The wave functions of QEs or QHs must be multi-component as any fractional statistics particles on a torus do [14, 15]. So it is a paradox. We will solve this paradox in this paper.

We organize the paper as follows: In the next section, we present a detailed discussion about constructing various hierarchical wave functions on a sphere (a brief discussion can be found in ref. [12]). Different hierarchical wave functions will be constructed and simplified. In section 3, we will review first what we have known about the hierarchical wave functions on a torus. Then we will present some new results, which include solving the puzzling of the multicomponent nature of the wave function of QEs and the construction of a hierarchical wave function without any singularity on a torus due to the condensation of QEs. Of course in studying the above mentioned problems, the open problem noted in ref. [3], the construction of a Laughlin wave function with QEs, would be addressed and solved.

2 Hierarchical wave functions on a sphere

In this section, we will discuss the construction of the hierarchical wave function on a sphere [2]. The quasiparticles satisfy fractional statistics [4], and the condensation of quasiparticles will give rise to the FQH state with

$$\nu = \frac{1}{p_1 + \frac{1}{p_2 + \frac{1}{\dots + \frac{1}{p_n}}}}, \quad (1)$$

where p_1 is an odd positive integer, and $p_i, i \neq 1$ are even integers (their signs are depend on the types of the quasiparticles, i.e., QH or QE excitations).

We use projective coordinates on the sphere (details of notations can be found in ref. [12]). The projective coordinates are given by $z = 2R \cot \frac{\theta}{2} e^{i\varphi}$ and its complex conjugate \bar{z} . We will take $R = 1/2$ for simplicity.

2.1 Quasiparticle excitations of a Laughlin state

The Laughlin wave function at filling $\nu = 1/m$ is

$$\Psi_m = \prod_{i < j}^N d(z_i, z_j)^m, \quad (2)$$

where

$$d(z_i, z_j) = \frac{z_i - z_j}{\sqrt{1 + z_i \bar{z}_i} \sqrt{1 + z_j \bar{z}_j}}. \quad (3)$$

The magnetic flux quanta ϕ out of the surface is equal to $m(N - 1)$ for the Laughlin state.

The Laughlin wave function with the presence of quasiparticle excitations is given by acting the quasiparticle excitation operators on the original Laughlin wave function. In the projective coordinates, the operators of the QH excitations and the quasielectron excitations are given in the following form,

$$A^\dagger(\omega_k, \bar{\omega}_k) \Psi_m(z_i) = \left[\prod_{j=1}^N \prod_{k=1}^{N_q} d(z_j, \omega_k) \right] \Psi_m(z_i), \quad (4)$$

$$A(\omega_k, \bar{\omega}_k) \Psi_m(z_i) = P(\phi) \left\{ \left[\prod_{j=1}^N \prod_{k=1}^{N_q} d(\bar{z}_j, \bar{\omega}_k) \right] \Psi_m(z_i) \right\} \quad (5)$$

where $\omega_k, \bar{\omega}_k$ are the coordinates of the quasiparticle. The flux ϕ in the presence of N_q QEs (QHs) is $m(N-1) - N_q$ ($m(N-1) + N_q$), and $P(\phi)$ is an operator which projects a state to the lowest Landau level with the magnetic flux quanta equal to ϕ [12]. Ref. [12] showed that the QE excitations given by eq. (5) is the same as that in ref. [2].

2.2 Laughlin wave function for quasiparticles

To construct a hierarchical wave function, we shall construct the wave function of quasiparticles.

First we consider a Laughlin state of electrons and its quasiparticles. When the quasiparticles condense, the wave function is of Jastrow (or Laughlin) type. The charge of a QH is $-e/m$ where e is the charge of an electron. As a QH has an opposite charge with respect to an electron, the wave function is anti-holomorphic with the coordinates of QHs. Because QHs satisfy fractional statistics with statistical parameter equal to $1/m$, the wave function for QHs of Laughlin type is found to be:

$$[\Psi_1(\bar{\omega}_k)]^{p+1/m}, \quad (6)$$

where Ψ_1 is defined as

$$\Psi_1(\omega_k) = \prod_{k < l}^{N_q} d(\omega_k, \omega_l). \quad (7)$$

The notation here is consistent with our definition of the Laughlin wave function.

QEs obey the same statistics as QHs. But as the charge of a QE is e/m , the wave function of QEs is holomorphic with the coordinates of QEs. From these facts, one can deduce that the wave function of QEs of Laughlin type is

$$[\Psi_1(\omega_k)]^{p-1/m} \quad (8)$$

We have constructed the wave function for quasiparticles of a Laughlin state at filling $\nu = 1/m$. The condensation of those quasiparticles lead to filling $\nu = 1/(m-1/p)$ or $\nu = 1/(m+1/p)$. Let us define the Laughlin state

at $\nu = 1/m$ is a first level hierarchical state. The quasiparticles of a Laughlin state can form a new Laughlin state of their own, and this will lead a second level hierarchical state as just discussed. The new Laughlin state then can have their own quasiparticles. If the new quasiparticles condense, it will lead to a third level hierarchical state. This process can continue, and the more complicated hierarchical states can be formed.

Suppose we have a Laughlin state formed by the quasiparticles at the n th level with θ_n as the statistical parameter. Our convention of the definition of θ_n here is: when we exchange the coordinates of two particles, we will get the same phase as we exchange two coordinates of the function $[\Psi_1(z_i^n)]^{-\theta_n}$. θ_n can be a positive or a negative rational number, but $|\theta_n|$ is generally less or equal than one.

If the charge of a quasiparticle is negative (we assume that the charge of an electron is negative), the wave function will take the following form,

$$[\Psi_1(z_i^n)]^{p_n - \theta_n} \quad (9)$$

where z_i^n is the coordinates of those quasiparticles. If the charge of the quasiparticle is positive, then the wave function will take the following form,

$$[\Psi_1(\bar{z}_i^n)]^{p_n + \theta_n}. \quad (10)$$

p_n is a positive even integer.

2.3 The excitations for the Laughlin state of quasiparticles

In order to construct the wave function of the quasiparticles at the $n+1$ th level, we need to know how to construct the wave function of quasiparticles of the n th level with their own quasiparticle excitations. We shall study the excitation of those states in the previous subsection. For the QH excitations of eq. (9) and eq. (10), the wave functions in the presence of those QHs will become

$$[\Psi_1(z_i^n)]^{p_n - \theta_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(z_j^n, z_k^{n+1}), \quad (11)$$

and

$$[\Psi_1(\bar{z}_i^n)]^{p_n+\theta_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(\bar{z}_j^n, \bar{z}_k^{n+1}), \quad (12)$$

where N_n is the number of quasiparticles at n th level, and z_i^n are the coordinates of those quasiparticles, z_i^{n+1} are the coordinates of new quasiparticles (we will denote the electron coordinates as z_i^1 in this notation) and N_{n+1} is the number of the new quasiparticles (N_1 is the number of electrons).

For the QE excitations of state (9) and state (10), the wave functions will be in the following forms;

$$[\Psi_1(z_i^n)]^{-\theta_n} P(\phi_n - N_{n+1}) \left\{ [\Psi_1(z_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(z_j^n, \bar{z}_k^{n+1}) \right\}, \quad (13)$$

and

$$[\Psi_1(\bar{z}_i^n)]^{\theta_n} P(\phi_n - N_{n+1}) \left\{ [\Psi_1(\bar{z}_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(z_j^n, z_k^{n+1}) \right\}, \quad (14)$$

where ϕ_n is equal to $N_n p_n - N_n$, and $P(\phi_n - N_{n+1})$ is an operator which projects the states to the lowest Landau levels with the magnetic flux $\phi_n - N_{n+1}$ (as previously defined). The construction of QEs in eq. 13 and eq. 14 is a natural generalization of the construction of QEs of the Laughlin state (see eq. (5)).

2.4 The normalizations of the wave functions

We need to normalize those states in the previous subsection for constructing the hierarchical wave function. The normalized states of eq. (11), eq. (12), eq. (13) and eq. (14) will be

$$[\Psi_1(z_i^n)]^{p_n-\theta_n} [\Psi_1(z_i^{n+1})]^{\frac{1}{p_n-\theta_n}} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(z_j^n, z_k^{n+1}), \quad (15)$$

$$[\Psi_1(\bar{z}_i^n)]^{p_n+\theta_n} [\Psi_1(\bar{z}_i^{n+1})]^{\frac{1}{p_n+\theta_n}} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(\bar{z}_j^n, \bar{z}_k^{n+1}), \quad (16)$$

$$[\Psi_1(z_i^n)]^{-\theta_n} [\Psi_1(z_i^{n+1})]^{\frac{1}{p_n-\theta_n}} \times P(\phi_n - N_{n+1}) \left\{ [\Psi_1(z_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(\bar{z}_j^n, \bar{z}_k^{n+1}) \right\}, \quad (17)$$

$$[\Psi_1(\bar{z}_i^n)]^{\theta_n} [\Psi_1(\bar{z}_i^{n+1})]^{\frac{1}{p_n+\theta_n}} \times P(\phi_n - N_{n+1}) \left\{ [\Psi_1(\bar{z}_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} d(z_j^n, z_k^{n+1}) \right\}. \quad (18)$$

The amplitudes of normalization factors can be determined by plasma analogue or by the rotational properties of the wave functions, and the phase of normalization factors are determined by the statistics of quasiparticles. The statistical parameters of θ_{n+1} for the quasiparticles z_k^{n+1} in eq. (11), eq. (12), eq. (13) and eq. (14) are respectively $\frac{1}{p_n-\theta_n}$, $-\frac{1}{p_n+\theta_n}$, $\frac{1}{p_n-\theta_n}$, and $-\frac{1}{p_n+\theta_n}$. With those parameters, we can build the Laughlin states for the news quasiparticles with coordinates z_k^{n+1} again, thus we can obtain the hierarchical wave function of any level.

2.5 Constructions of hierarchical wave functions on a sphere

We finally come to construct hierarchical wave functions. We first consider when ν is given by

$$\frac{1}{p_1 - \frac{1}{p_2 - \frac{1}{\dots - \frac{1}{p_n}}}} \quad (19)$$

with p_i being positive integers. This state is obtained when the quasiparticles in any level are of QE type. Most of observed fillings in experiments can be given by eq. (19).

Take $n = 3$, then the wave function is given as

$$\int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{\beta=1}^{N_3} [dz_{\beta}^2] \Psi^1 \Psi^2 \Psi^3, \quad (20)$$

where we define that $[dz_{\alpha}^i]$ is equal to $\frac{dz_{\alpha}^i d\bar{z}_{\alpha}^i}{(1+z_{\alpha}^i \bar{z}_{\alpha}^i)^2}$, and Ψ^1 as the normalized wave function of electrons with quasielectron excitations, Ψ^2 as the normalized wave function of quasiparticles of the electron state, etc.. By using formulas

previously obtained in the last subsection, we can write the wave function explicitly,

$$\begin{aligned}
\Psi = & \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{\beta=1}^{N_3} [dz_{\beta}^3] P(p_1 N_1 - p_1 - N_2, z_i^1) \\
& \times [\Psi_1(z_i^1)]^{p_1} \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i^1, \bar{z}_{\alpha}^2) [\Psi_1(z_i^2)]^{1/p_1} \\
& \times [\Psi_1(z_i^2)]^{-1/p_1} P(p_2 N_2 - p_2 - N_3, z_i^2) \\
& \times [\Psi_1(z_i^2)]^{p_2} \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_{\alpha}^3) [\Psi_1(z_i^3)]^{\frac{1}{p_2-1/p_1}} \\
& \times [\Psi_1(z_i^3)]^{\frac{-1}{p_2-1/p_1}} [\Psi_1(z_i^3)]^{p_3}, \tag{21}
\end{aligned}$$

and which can simplify to

$$\begin{aligned}
\Psi = & \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{\beta=1}^{N_3} [dz_{\beta}^3] P(p_1 N_1 - p_1 - N_2, z_i^1) \\
& \times [\Psi_1(z_i^1)]^{p_1} \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i^1, \bar{z}_{\alpha}^2) P(p_2 N_2 - p_2 - N_3, z_i^2) \\
& \times [\Psi_1(z_i^2)]^{p_2} \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_{\alpha}^3) [\Psi_1(z_i^3)]^{p_3}. \tag{22}
\end{aligned}$$

We can further simplify the wave function. The key observation is that the following equation holds

$$\langle \psi_1 | P | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle \tag{23}$$

where ψ_1 is a state in the lowest Landau level, and P is an operator which projects ψ_2 to the lowest Landau Level. In eq. (22). We consider only the part of function which depends on z_i^2 ,

$$\begin{aligned}
& \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i^1, \bar{z}_{\alpha}^2) \times \\
& P(p_2 N_2 - p_2 - N_3, z_i^2) [\Psi_1(z_i^2)]^{p_2} \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_{\alpha}^3), \tag{24}
\end{aligned}$$

which is equal to the inner product of two bosonic wave functions as in eq. (23), when

$$\psi_1 = \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i^1, \bar{z}_\alpha^2)$$

and

$$\left[\Psi_1(z_i^2) \right]^{p_2} \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_\alpha^3).$$

Thus we can simply omit $P(p_2 N_2 - p_2 - N_3, z_i^2)$ in eq. (24) and also in eq. (22). Thus we can write the wave function in a simple form,

$$\begin{aligned} \Psi = & P(\phi, z_i^1) \int \prod_{\alpha=1}^{N_2} [dz_\alpha^2] \prod_{\beta=1}^{N_3} [dz_\beta^3] \\ & \times \left[\Psi_1(z_i^1) \right]^{p_1} \left[\Psi_1(z_i^2) \right]^{p_2} \left[\Psi_1(z_i^3) \right]^{p_3} \\ & \times \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i^1, \bar{z}_\alpha^2) \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_\alpha^3) \end{aligned} \quad (25)$$

where $\phi = p_1 N_1 - p_1 - N_2$ is the magnetic flux out of the sphere (note again that z_i^1 are actually the coordinates of electrons).

We can construct the n level hierarchical wave function with filling given by eq. (19), and then simplify it following previous discussions about the case of $n = 3$. The wave function is

$$\begin{aligned} \Psi = & P(\phi, z_i^1) \int \prod_{l=2}^n \prod_{k=1}^{N_l} [dz_k^l] \prod_{l=1}^n \left[\Psi_1(z_i^l) \right]^{p_l} \\ & \times \prod_{l=1}^{n-1} \prod_{i=1}^{N_l} \prod_{j=1}^{N_{l+1}} d(\bar{z}_i^l, \bar{z}_j^{l+1}). \end{aligned} \quad (26)$$

The hierarchical wave function due to the condensations of quasielectrons is amazingly simple, and it could be calculated analytically.

Ref. [8] had proposed a different hierarchical wave function with the same filling as the hierarchical wave function of eq. (26). The wave function is

$$\begin{aligned} \Psi = & \int \prod_{l=2}^n \prod_{k=1}^{N_l} [dz_k^l] \prod_{l=1}^n \left[\Psi_1(z_i^l) \right]^{p_l} \\ & \times \prod_{l=1}^{n-1} \prod_{i=1}^{N_l} \prod_{j=1}^{N_{l+1}} \frac{1}{d(z_i^l, z_j^{l+1})}. \end{aligned} \quad (27)$$

However eq. (27) contains singularities and we do not know how to control them if we want to carry out the calculation of eq. (27). If we do the integration in eq. (27), the wave function will not be holomorphic because of those singularities, thus we need also to project the wave function of eq. (27) to the lowest Landau level after the integration.

We could derive eq. (27) by assuming that the Laughlin wave function with QE excitations is

$$A(\omega_k, \bar{\omega}_k) \Psi_m(z_i) = \left\{ \left[\prod_{j=1}^N \prod_{k=1}^{N_q} \frac{1}{d(z_j, \omega_k)} \right] \Psi_m(z_i) \right\} \quad (28)$$

instead of eq. (5), and similarly eq. (13) and eq. (14) are replaced by the following equations,

$$[\Psi_1(z_i^n)]^{-\theta_n} \left\{ [\Psi_1(z_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} \frac{1}{d(z_j^n, z_k^{n+1})} \right\},$$

and

$$[\Psi_1(\bar{z}_i^n)]^{\theta_n} \left\{ [\Psi_1(\bar{z}_i^n)]^{p_n} \prod_{j=1}^{N_n} \prod_{k=1}^{N_{n+1}} \frac{1}{d(\bar{z}_j^n, \bar{z}_k^{n+1})} \right\}.$$

Of course we know that those constructions for QEs are not favorable as the previous constructions given by eq. (5), eq. (13) and eq. (14).

All wave functions in the FQHE on the sphere shall be rotationally invariant. Apply this condition for the wave functions of eq. (26) and eq. (27), we obtain

$$\left[\sum_{j=1}^n \Lambda_{ij} N_j \right] - \Lambda_{ii} = \begin{cases} \phi, & \text{if } i = 1; \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

where

$$\Lambda = \begin{pmatrix} p_1 & -1 & 0 & \dots & 0 & 0 \\ -1 & p_2 & -1 & 0 & \dots & 0 \\ 0 & -1 & p_3 & -1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & p_{n-1} & -1 \\ 0 & 0 & \dots & 0 & -1 & p_n \end{pmatrix}. \quad (30)$$

We will now discuss the case when the filling is given by eq. (1) with p_i all positive. When $n = 3$, the wave function is

$$\begin{aligned} \Psi = & \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{\beta=1}^{N_3} [dz_{\beta}^3] [\Psi_1(z_i^1)]^{p_1} \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(z_i^1, z_{\alpha}^2) [\Psi_1(z_i^2)]^{1/p_1} \\ & \times [\Psi_1(\bar{z}_i^2)]^{1/p_1} [\Psi_1(\bar{z}_i^2)]^{p_2} \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_{\alpha}^3) [\Psi_1(\bar{z}_i^3)]^{\frac{1}{p_2+1/p_1}} \\ & \times [\Psi_1(z_i^3)]^{\frac{1}{p_2+1/p_1}} [\Psi_1(z_i^3)]^{p_3}, \end{aligned} \quad (31)$$

and which can be written as

$$\begin{aligned} \Psi = & \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{\beta=1}^{N_3} [dz_{\beta}^3] [\Psi_1(z_i^1)]^{p_1} [\Psi_1(\bar{z}_i^2)]^{p_2} [\Psi_1(z_i^3)]^{p_3} \\ & \times \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(z_i^1, z_{\alpha}^2) \prod_{i=1}^{N_2} \prod_{\alpha=1}^{N_3} d(\bar{z}_i^2, \bar{z}_{\alpha}^3) \\ & \times [\Psi_1(z_i^2) \Psi_1(\bar{z}_i^2)]^{1/p_1} [\Psi_1(z_i^3) \Psi_1(\bar{z}_i^3)]^{\frac{1}{p_2+1/p_1}}. \end{aligned} \quad (32)$$

The hierarchical wave function due to the condensations of quasiholes now appears more complicated and it can not be integrated analytically. For any n , the wave function is

$$\begin{aligned} \Psi = & \int \prod_{l=2}^n \prod_{\alpha=1}^{N_l} [dz_{\alpha}^l] \prod_{1 \leq 2l-1 \leq n} [\Psi_1(z_i^{2l-1})]^{p_{2l-1}} \prod_{1 \leq 2l \leq n} [\Psi_1(\bar{z}_i^{2l})]^{p_{2l}} \\ & \times \prod_{1 \leq 2l-1 \leq n} \prod_{i=1}^{N_{2l-1}} \prod_{\alpha=1}^{N_{2l}} d(z_i^{2l-1}, z_{\alpha}^{2l}) \prod_{1 \leq 2l \leq n} \prod_{i=1}^{N_{2l}} \prod_{\alpha=1}^{N_{2l+1}} d(\bar{z}_i^{2l}, \bar{z}_{\alpha}^{2l+1}) \\ & \times \prod_{l=2}^n [\Psi_1(z_i^l) \Psi_1(\bar{z}_i^l)]^{\epsilon_l} \end{aligned} \quad (33)$$

where ϵ_l is determined by the relation

$$\epsilon_{l+1} = \frac{1}{p_l + \epsilon_l}, \epsilon_l = 0. \quad (34)$$

In refs. [12, 16], it was argued that eq. (33) can be approximated by omitting $[\Psi_1(z_i^l) \Psi_1(\bar{z}_i^l)]^{\epsilon_l}$ in the integration,

$$\Psi = \int \prod_{l=2}^n \prod_{\alpha=1}^{N_l} [dz_{\alpha}^l] \prod_{1 \leq 2l-1 \leq n} [\Psi_1(z_i^{2l-1})]^{p_{2l-1}} \prod_{1 \leq 2l \leq n} [\Psi_1(\bar{z}_i^{2l})]^{p_{2l}}$$

$$\times \prod_{1 \leq 2l-1 \leq n} \prod_{i=1}^{N_{2l-1}} \prod_{\alpha=1}^{N_{2l}} d(z_i^{2l-1}, z_\alpha^{2l}) \prod_{1 \leq 2l \leq n} \prod_{i=1}^{N_{2l}} \prod_{\alpha=1}^{N_{2l+1}} d(\bar{z}_i^{2l}, \bar{z}_\alpha^{2l+1}). \quad (35)$$

The wave function of eq. (35) was also found to be a good trial wave function comparing the exact ground state of a small number of electrons. Apply the rotational invariance condition on the wave functions of eq. (33) and eq. (35), one finds again that eq. (29) must be satisfied, however the matrix Λ is now given by

$$\Lambda = \begin{pmatrix} p_1 & +1 & 0 & \dots & 0 & 0 \\ +1 & -p_2 & -1 & 0 & \dots & 0 \\ 0 & -1 & p_3 & +1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & (-1)^{n-1} & (-1)^n p_{n-1} & (-1)^n \\ 0 & 0 & \dots & 0 & (-1)^n & (-1)^{n+1} p_n \end{pmatrix}. \quad (36)$$

We have discussed hierarchical wave functions due to the QE condensation or due to the QH condensations. Experimentally, hierarchical states due to the condensation of both QEs and QHs were also observed, for example, $\nu = n/(4n-1)$ with $n \geq 3$. $\nu = n/(4n-1)$ can be written as

$$\frac{1}{p_1 + \frac{1}{p_2 - \frac{1}{\dots - \frac{1}{p_n}}}} \quad (37)$$

with $p_1 = 3$, $p_i = 2, i \neq 1$. The wave function at ν of eq. (37) can be constructed straight forwardly. After the simplifications, the wave function is

$$\begin{aligned} \Psi = & \left[\Psi_1(z_i^1) \right]^{p_1} \int \prod_{k=1}^{N_2} [dz_k^2] \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} d(z_i^1, z_j^2) \left[\Psi_1(z_i^2) \Psi_1(\bar{z}_i^2) \right]^{1/p_1} \\ & P(p_2 N_2 - p_2 - N_3, z_i^2) \int \prod_{l=3}^n \prod_{k=1}^{N_l} [dz_k^l] \prod_{l=2}^n \left[\Psi_1(\bar{z}_i^l) \right]^{p_l} \\ & \times \prod_{l=2}^{n-1} \prod_{i=1}^{N_l} \prod_{j=1}^{N_{l+1}} d(z_i^l, z_j^{l+1}). \end{aligned} \quad (38)$$

We could approximate the wave function by taking out $[\Psi_1(z_i^2)\Psi_1(\bar{z}_i^2)]^{1/p_1}$ (note that all wave functions discussed in the paper are trial wave functions, not the exact wave function), the wave function can be then written as

$$\begin{aligned} \Psi = & \left[\Psi_1(z_i^1) \right]^{p_1} \int \prod_{k=1}^{N_2} [dz_k^2] \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} d(z_i^1, z_j^2) \times \\ & \int \prod_{l=3}^n \prod_{k=1}^{N_l} [dz_k^l] \prod_{l=2}^n \left[\Psi_1(\bar{z}_i^l) \right]^{p_l} \prod_{l=2}^{n-1} \prod_{i=1}^{N_l} \prod_{j=1}^{N_{l+1}} d(z_i^l, z_j^{l+1}). \end{aligned} \quad (39)$$

Now we apply the rotational invariance conditions on the wave functions of eq. (38) and eq. (39). Again eq. (29) must be satisfied, and the matrix Λ is now,

$$\Lambda = \begin{pmatrix} p_1 & +1 & 0 & \dots & 0 & 0 \\ +1 & -p_2 & +1 & 0 & \dots & 0 \\ 0 & +1 & -p_3 & +1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & +1 & -p_{n-1} & +1 \\ 0 & 0 & \dots & 0 & +1 & -p_n \end{pmatrix}. \quad (40)$$

In this section, we construct various hierarchical wave functions. In the case of the hierarchical wave function with the filling given by eq. (19), we greatly simplified the wave function and obtained a very simple formula for the wave function. In other cases, we can also approximate the wave functions and the resulted wave functions also appear in quite simple forms.

3 Hierarchical wave functions on a torus

All the wave functions on the sphere constructed in the last section can be generalized to the case when the space is a torus. However the construction of the similar wave functions on a torus is much more complicated than on a sphere. The wave function of a Laughlin wave function on a torus with QE excitation has been unknown and the construction of such wave functions has remained as an open question. We will solve this open question in this section and discuss in detail the construction of hierarchical wave functions on a torus.

3.1 Quasiparticles on a torus

The Laughlin wave function on a torus was obtained in ref. [3] and was reformulated in refs. [13, 10], which we will follow (see also the appendix).

The (normalized) Laughlin wave function of electrons with QH excitations on a torus is

$$\begin{aligned}\Psi^1(z_i^1, z_\alpha^2) &= \exp\left(-\frac{\pi\phi(\sum_i [y_i^1]^2 + \frac{1}{p_1} \sum_\alpha [y_\alpha^2]^2)}{\tau_2}\right) F^1(z_i^1, z_\alpha^2), \\ F^1(z_i^1, z_\alpha^2) &= \theta\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right] \left(\sum_i z_i^1 e_1 + \sum_\alpha z_\alpha^2 e_1^* | e_1, \tau\right) \prod_{i < j}^{N_1} [\theta_3(z_i^1 - z_j^1 | \tau)]^{p_1} \\ &\quad \times \prod_{i, \alpha}^{N_1, N_2} [\theta_3(z_i^1 - z_\alpha^2 | \tau)] \prod_{\alpha < \beta}^{N_2} [\theta_3(z_\alpha^2 - z_\beta^2 | \tau)]^{\frac{1}{p_1}},\end{aligned}\quad (41)$$

where $e_1^2 = p_1$, $e_1^* = \frac{1}{e}$ and a_1, b_1 are still given by eq. (80), e.g. $a_1^* = a_0 + l$, $b_1^* = b_0$ (a_1, b_1 are determined by the boundary conditions), and magnetic flux $\phi = p_1 N_1 + N_2$.

Now we want to construct the QE excitations on a torus. Before give the correct construction, we will construct the QE excitations on a torus with singularities similar to the QE excitations on a sphere with singularities given by eq. (28):

$$\begin{aligned}\Psi^1(z_i^1, z_\alpha^2) &= \exp\left(-\frac{\pi\phi(\sum_i [y_i^1]^2 - \frac{1}{p_1} \sum_\alpha [y_\alpha^2]^2)}{\tau_2}\right) F^1(z_i^1, z_\alpha^2), \\ F^1(z_i^1, z_\alpha^2) &= \theta\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right] \left(\sum_i z_i^1 e_1 - \sum_\alpha z_\alpha^2 e_1^* | e_1, \tau\right) \prod_{i < j} [\theta_3(z_i^1 - z_j^1 | \tau)]^{p_1} \\ &\quad \times \left(\prod_{i, \alpha} [\theta_3(z_i^1 - z_\alpha^2 | \tau)] \prod_{\alpha < \beta}\right)^{-1} [\theta_3(z_\alpha^2 - z_\beta^2 | \tau)]^{\frac{1}{p_1}},\end{aligned}\quad (42)$$

with a_1, b_1 given by eq. (80) again, and $\phi = p_1 N_1 - N_2$.

In order to construct hierarchical wave functions on a torus, we shall generalize the quasiparticle wave function of Laughlin type on a sphere to the one on a torus.

Suppose that a wave function on a sphere is $\prod_{i < j}^N [d(z_i, z_j)]^{p/q}$ with p and q being coprime to each other. We find that, on a torus, the wave function

with N_q QH excitations would be [13]:

$$\begin{aligned}\Psi(z_i, \omega_\alpha) &= \exp\left(-\frac{\pi(pN + qN_q)(p\sum_i[y_i]^2 + q\sum_\alpha[y_\alpha(\omega)]^2)}{pq\tau_2}\right) F(z_i, \omega_\alpha), \\ F(z_i, \omega_\alpha) &= \theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \left(\sum_i z_i s + \sum_\alpha \omega_\alpha s^* |e, \tau\right) \prod_{i < j} [\theta_3(z_i - z_j | \tau)]^{p/q} \\ &\quad \times \prod_{i, \alpha} [\theta_3(z_i - \omega_\alpha | \tau)] \prod_{\alpha < \beta} [\theta_3(\omega_\alpha - \omega_\beta | \tau)]^{q/p},\end{aligned}\quad (43)$$

and the wave function with N_q QE excitations (of the type which contains singularities) :

$$\begin{aligned}\Psi(z_i, \omega_\alpha) &= \exp\left(-\frac{\pi(pN - qN_q)(p\sum_i[y_i]^2 - q\sum_\alpha[y_\alpha(\omega)]^2)}{pq\tau_2}\right) F(z_i, \omega_\alpha), \\ F(z_i, \omega_\alpha) &= \theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \left(\sum_i z_i s - \sum_\alpha \omega_\alpha s^* |e, \tau\right) \prod_{i < j} [\theta_3(z_i - z_j | \tau)]^{p/q} \\ &\quad \times \left(\prod_{i, \alpha} [\theta_3(z_i - \omega_\alpha | \tau)]\right)^{-1} \prod_{\alpha < \beta} [\theta_3(\omega_\alpha - \omega_\beta | \tau)]^{q/p},\end{aligned}\quad (44)$$

with $s = (p/q)^{1/2}$, $s^* = (q/p)^{1/2}$, $e = (pq)^{1/2}$. $a = a^*e^*$, $b = b^*e^*$ with $e^* = 1/e$ can be determined by the boundary conditions. They can be written as (of course there are other ways to write out the solutions as we will see later on):

$$\begin{aligned}a^* &= a_0 + p\lambda_1 + q\lambda_2, b^* = b_0, \\ \lambda_1 &= 1, \dots, q, \lambda_2 = 1, \dots, p,\end{aligned}\quad (45)$$

with a_0, b_0 fixed by the boundary conditions. We will denote the wave function Ψ as $\Psi(z_i, \omega_\alpha | \lambda_1, \lambda_2)$. For z_i particles, the wave function has λ_2 degeneracies and λ_1 components (the wave function is represented by a column, not a single function). For ω_i particles, the wave function has λ_2 degeneracies and λ_1 components. This fact represents the particle-vortex duality in the system.

The wave functions of eq. (43) and eq. (44) satisfy the braid group relations required for anyons (particles obeying fractional statistics) on a torus [14, 15].

We can use eq. (43) and eq. (44) to obtain quasiparticle wave functions on a torus. Remind that for the wave function with QE excitations, eq. (44),

the wave function contains singularities and we will return to this point later on.

3.2 Constructions of hierarchical wave functions on a torus

We will use eq. (43) and eq. (44) to construct the hierarchical wave function on a torus. When ν is given by eq. (1) with p_i all positive integers, all quasiparticles are of QH type, the wave function was obtained in ref. [13], and we will not repeat its derivation here.

Generally, the wave function of the n level hierarchy on a torus can be written as

$$\int \prod_{l=2}^n \prod_{\alpha=1}^{N_l} [dz_{\alpha}^l] \sum_{\lambda_1, \dots, \lambda_{n-1}} \Psi^1(z_i^1, z_{\alpha}^2 | \lambda_1) \times \dots \times \Psi^l(z_i^l, z_{\alpha}^{l+1} | \lambda_{l-1}, \lambda_l) \dots \times \Psi^n(z_i^n | \lambda_{n-1}, \lambda_n) \quad (46)$$

with $[dz_{\alpha}^l] = dz_{\alpha}^l d\bar{z}_{\alpha}^l$. We obtain a wave function with a degeneracy index λ_n .

We consider the case when ν is given by eq. (19). First we take the simplest case, $n = 2$. Following eq. (46), the wave function would be:

$$\begin{aligned} \Psi(z_i^1) &= \exp\left(-\frac{\pi\phi \sum_i [y_i^1]^2}{\tau_2}\right) \int \prod_{\alpha=1}^{N_2} [dz_{\alpha}^2] \prod_{i < j} [\theta_3(z_i^1 - z_j^1 | \tau)]^{p_1} \\ &\times \left(\prod_{i, \alpha} [\theta_3(z_i^1 - z_{\alpha}^2 | \tau)] \right)^{-1} \prod_{\alpha < \beta} [\theta_3(z_{\alpha}^2 - z_{\beta}^2 | \tau)]^{p_2} \\ &\times \sum_{\lambda_1} \Theta^1(z_i^1, z_{\alpha}^2 | \lambda_1) \Theta^2(z_i^2 | \lambda_1, \lambda_2) \end{aligned} \quad (47)$$

where

$$\begin{aligned} \Theta^1(z_i^1, z_{\alpha}^2 | \lambda_1) &= \theta \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \left(\sum_i z_i^1 s_1 - \sum_{\alpha} z_{\alpha}^2 s_1^* | e_1, \tau \right), \\ a_1^* &= a_0 + \lambda_1, \lambda_1 = 1, \dots, p_1, b_1^* = b_0, \\ s_1 &= (p_1)^{1/2}, s_1^* = \frac{1}{(p_1)^{1/2}}, e_1 = (p_1)^{1/2}, \end{aligned} \quad (48)$$

and

$$\begin{aligned}
\Theta^2(z_\alpha^2|\lambda_1, \lambda_2) &= \theta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \left(\sum_\alpha z_\alpha^2 s_2 |e_2, \tau \right), \\
a_2^* &= a_0 - \lambda_1(p_1 p_2 - 1) + \lambda_2, b_2^* = b_0 \\
\lambda_1 &= 1, \dots, p_1, \lambda_2 = 1, \dots, (p_1 p_2 - 1), \\
s_2 &= (p_2 - \frac{1}{p_1})^{1/2}, e_2 = [p_1(p_1 p_2 - 1)]^{1/2},
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
p_1 N_1 - N_2 &= \phi, \\
N_1 - p_2 N_2 &= 0,
\end{aligned} \tag{50}$$

with a_0, b_0 given by eq. (80). a_1, b_1 are determined by the boundary conditions for electrons, a_2, b_2 are determined by requiring the function within the integration of eq. (47) being periodic with coordinates z_i^2 . Those boundary conditions also lead to eq. (50).

λ_2 is the degeneracy index and the degeneracy is thus equal to $p_1 p_2 - 1$, agreed with the general results obtained in ref. [17].

We can rewrite

$$\sum_{\lambda_1} \Theta^1(z_i^1, z_\alpha^2|\lambda_1) \Theta^2(z_i^2|\lambda_1, \lambda_2) \tag{51}$$

as a theta function on a two dimensional lattice [18]. One can show that a theta function on a two dimensional lattice would satisfy the same translational properties as the function given by eq. (51). The theta function on a two dimensional lattice is

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|e, \tau) = \sum_{n_i} \exp(\pi i (v + a)^2 \tau + 2\pi i (v + a) \cdot (z + b)),$$

with

$$\begin{aligned}
e_i \cdot e_j &= A = \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix}, z = \sum_i z_i^2 e_1 + \sum_\alpha z_\alpha^2 e_2, \\
a &= [\frac{\phi_1}{2\pi} + \frac{\Phi + 1}{2}] e_1^* + \sum_s n^s e_s^*, b = [-\frac{\phi_2}{2\pi} + \frac{\Phi + 1}{2}] e_1^*, \\
n^s e_s^* &\subset \frac{\Lambda^*}{\Lambda}.
\end{aligned} \tag{52}$$

e_i defined here as a vector in a lattice should not be confused with e_i defined in eq. (48) and eq. (49), we are sorry for the abuse of notations.

The linear independent theta functions appeared in eq. (52) are given by the independent vectors in $n^s e_s^* \subset \frac{\Lambda^*}{\Lambda}$, and the number of such independent vectors is simply $\det A = p_1 p_2 - 1$.

Thus we conclude that the linear independent functions given by eq. (51) can be expressed by the linear independent functions given by eq. (52). The discussion of such mathematical relations can be also found in ref. [18]. Using this result, the wave function of eq. (47) can be written as

$$\begin{aligned} \Psi(z_i^1) = & \exp\left(-\frac{\pi\phi \sum_i [y_i^1]^2}{\tau_2}\right) \int \prod_{\alpha=1}^{N_2} [dz_\alpha^2] \prod_{i<j} [\theta_3(z_i^1 - z_j^1|\tau)]^{p_1} \\ & \times \left(\prod_{i,\alpha} [\theta_3(z_i^1 - z_\alpha^2|\tau)] \right)^{-1} \prod_{\alpha<\beta} [\theta_3(z_\alpha^2 - z_\beta^2|\tau)]^{p_2} \\ & \times \theta \begin{bmatrix} a \\ b \end{bmatrix} \left(\sum_i z_i^1 e_1 + \sum_\alpha z_\alpha^2 e_2 | e, \tau \right) \end{aligned} \quad (53)$$

with a, b, e_i given by eq. (52).

Remarkably, the wave function of eq. (53) had been already obtained in refs. [10, 11]. Here we give a derivation of the wave function of eq. (53) based on the picture of the hierarchical theory. It was quite puzzling that the wave function of eq. (53) does not show that the wave function of quasiparticles is a multicomponent wave function, in contrary to the wave function obtained in ref. [13] when ν is given by eq. (1). Now the puzzle is solved because the wave function of eq. (53) is equivalent to the wave function of eq. (47), and eq. (47) shows that the wave function of quasiparticles is a multicomponent wave function.

We can repeat the discussion for an arbitrary n , generalizing the wave function of eq. (47) to the one for the n level hierarchy, and then simplifying it by using the relations of the theta function on a n dimensional lattice, we will get:

$$\begin{aligned} \Psi(z_i^1) = & \exp\left(-\frac{\pi\phi \sum_i [y_i^1]^2}{\tau_2}\right) \int \prod_{l=2}^n \prod_{\alpha=1}^{N_l} [dz_\alpha^l] \\ & \prod_{l=1}^n \prod_{i<j} [\theta_3(z_i^l - z_j^l|\tau)]^{p_l} \prod_{l=1}^{n-1} \left(\prod_{i,\alpha}^{N_l, N_{l+1}} [\theta_3(z_i^l - z_\alpha^{l+1}|\tau)] \right)^{-1} \end{aligned}$$

$$\times \theta \begin{bmatrix} a \\ b \end{bmatrix} (\sum_{i,l} z_i^l e_l |e, \tau) \quad (54)$$

with

$$\begin{aligned} e_i \cdot e_j = A &= \begin{pmatrix} p_1 & -1 & 0 & \dots & 0 & 0 \\ -1 & p_2 & -1 & 0 & \dots & 0 \\ 0 & -1 & p_3 & -1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & p_{l-1} & -1 \\ 0 & 0 & \dots & 0 & -1 & p_l \end{pmatrix}, \\ a &= [\frac{\phi_1}{2\pi} + \frac{\Phi+1}{2}]e_1^* + \sum_s n^s e_s^*, b = [-\frac{\phi_2}{2\pi} + \frac{\Phi+1}{2}]e_1^*, \\ n^s e_s^* &\subset \frac{\Lambda^*}{\Lambda}, \\ A_{ij} N_j &= \phi \delta_{i,1}. \end{aligned} \quad (55)$$

Because the wave function contains singularities, the wave function does not represent a state on the lowest Landau level and one needs to project the wave function to the lowest Landau level in the end. The degeneracy is given the number of independent vectors in $n^s e_s^* \subset \frac{\Lambda^*}{\Lambda}$ and it is equal to $\det A$.

3.3 Quasielectrons on a torus

We note that the QE wave function used in our construction of the wave function of eq. (54) contains singularities. We will construct a Laughlin wave function with QE excitations on a torus without singularities, which was mentioned as an open problem in ref. ([3]) and still is an open problem today.

The Laughlin wave function with QE excitations on a sphere can be written as a derivative operator acts on the Laughlin wave function [2]. It is very difficult to generalize this derivative operator on a sphere to an operators on a torus as suggested in ref. [3]. However the Laughlin wave function with QE excitations on a sphere can be also written as the projection to the lowest Landau level by a wave function which contains higher Landau level states as described in eq. (5). We will find that it is actually quite **easy** to generalize

eq. (5) to the case on a torus. What we need to do is to replace

$$\left(\prod_{i,\alpha} [\theta_3(z_i - \omega_\alpha | \tau)] \right)^{-1}$$

in eq. (41) by a function, which is regular, satisfies the same translational properties (which must hold for any trial wave function), and is not necessary a holomorphic function.

The important observation is that the function

$$\overline{\theta_3(z|\tau)} \exp \left[\frac{\pi(z - \bar{z})^2}{2\tau_2} \right]$$

with $\overline{\theta_3(z|\tau)}$ as the complex conjugate of $\theta_3(z|\tau)$ has the same translational properties as

$$\frac{1}{\theta_3(z|\tau)}.$$

Thus we propose that the Laughlin wave function with QE excitations on a torus is

$$\begin{aligned} \Psi^1(z_i^1, z_\alpha^2) &= P(\phi, z_i^1) \exp\left(-\frac{\pi\phi(\sum_i [y_i^1]^2 - \frac{1}{p_1} \sum_\alpha [y_\alpha^2]^2)}{\tau_2}\right) F^1(z_i^1, z_\alpha^2), \\ F^1(z_i^1, z_\alpha^2) &= \theta \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \right] \left(\sum_i z_i^1 e_1 - \sum_\alpha z_\alpha^2 e_1^* | e_1, \tau \right) \prod_{i < j} [\theta_3(z_i^1 - z_j^1 | \tau)]^{p_1} \\ &\quad \times \prod_{i,\alpha} \overline{\theta_3(z_i^1 - z_\alpha^2 | \tau)} \exp \left[\frac{\pi(z_i^1 - z_\alpha^2 - \bar{z}_i^1 + \bar{z}_\alpha^2)^2}{2\tau_2} \right] \\ &\quad \times \prod_{\alpha < \beta} [\theta_3(z_\alpha^2 - z_\beta^2 | \tau)]^{\frac{1}{p_1}}, \end{aligned} \quad (56)$$

where $P(\phi, z_i^1)$ with $\phi = p_1 N_1 - N_2$ is an operator which projects the wave function

$$\exp\left(-\frac{\pi\phi(\sum_i [y_i^1]^2 - \frac{1}{p_1} \sum_\alpha [y_\alpha^2]^2)}{\tau_2}\right) F^1(z_i^1, z_\alpha^2) \quad (57)$$

to the lowest Landau level. All parameters in eq. (56) are the same as those in eq. (42).

One can check the wave function of eq. (57) satisfies the translational properties required for the electrons on a torus with magnetic flux $\phi = p_1 N_1 - N_2$ (see the appendix). However the state represented by the wave function of eq. (57) does not lie purely on the lowest Landau level and one needs to project it to the lowest Landau level in the end to obtain the Laughlin wave function with QE excitations on a torus. We note that eq. (56) is a direct generalization of eq. (5) which is a construction of the similar wave function on a sphere. In the case of eq. (5), one can replace the projection operator and function $d(\bar{z}_j, \bar{\omega}_k)$ by a derivative operator. However in the case of eq. (56), one can **not** find a similar derivative operator for constructing QE excitations on a torus as in the case of a sphere (at least we do not know how to do that now). It seems that the construction of QE excitations involves intrinsically higher Landau levels on a torus. The construction on the torus can also be generalized to the case when the surface is a high genus Riemann surface or other complicated surfaces (the Laughlin wave function on a Riemann surface was obtained in ref. ([19])). We finally comment that it seems that the construction of the Laughlin wave function with QE excitations given by eq. (56) is quite unique and we do not know any other plausible constructions. Such construction by eq. (56) could (and should) be checked by numerical calculations.

We can also construct QE excitations for the Laughlin wave function of quasiparticles. We will not discuss it here, and we comment that it involves more deep understandings of the singular gauge for anyons on a torus (see ref. [20] for the detailed discussions of the singular gauge on a torus).

3.4 Hierarchical wave functions on a torus revisited

We can now construct the hierarchical wave function on a torus by using the construction of QE excitations discussed in the previous subsection. The part of the wave function which is dependent on the center coordinates is the same as the one we obtained by using the construction of QE excitations with singularities. We have a trick to derive the wave function without doing similar calculations done in the previous subsections. The trick is that we simply replace all functions like $(\theta_3(z))^{-1}$ by $\overline{\theta_3(z)} \exp \frac{\pi(z-\bar{z})^2}{2\tau_2}$ in the wave

function of eq. (54), and we get

$$\begin{aligned}
\Psi(z_i^1) &= P(\phi, z_i^1) \exp\left(-\frac{\pi\phi \sum_i [y_i^1]^2}{\tau_2}\right) \int \prod_{l=2}^n \prod_{\alpha=1}^{N_l} [dz_\alpha^l] \\
&\times \prod_{l=1}^n \prod_{i < j} [\theta_3(z_i^l - z_j^l | \tau)]^{p_l} \prod_{l=1}^{n-1} \prod_{i, \alpha}^{N_l, N_{l+1}} \overline{\theta_3(z_i^l - z_\alpha^{l+1} | \tau)} \\
&\times \exp \frac{\pi[z_i^l - z_\alpha^{l+1} - \bar{z}_i^l + \bar{z}_\alpha^{l+1}]^2}{2\tau_2} \theta \begin{bmatrix} a \\ b \end{bmatrix} \left(\sum_{i,l} z_i^l e_l | e, \tau \right), \quad (58)
\end{aligned}$$

with all parameters given by eq. (55).

Now we show an example when ν is given by eq. (1) (detailed discussions can be found in ref. [13]). The wave function at $\nu = \frac{1}{p_1+1/p_2}$ is

$$\int \prod_{\alpha=1}^{N_2} [dz_\alpha^2] \sum_{\lambda_1} \Psi^1(z_i^1, z_\alpha^2 | \lambda_1) \Psi^2(z_\alpha^2 | \lambda_1, \lambda_2)$$

where $\Psi^1(z_i^1, z_\alpha^2 | \lambda_1)$ is given by eq. (41) with $a^* = a_0 + \lambda_1$, and $\Psi^2(z_\alpha^2 | \lambda_1, \lambda_2)$ is given by the following equation,

$$\begin{aligned}
\Psi^2(z_\alpha^2 | \lambda_1, \lambda_2) &= \exp\left[-\frac{\pi\phi \sum_\alpha (y_\alpha^2)^2}{p_1\tau_2}\right] F^2(z_\alpha^2 | \lambda_1, \lambda_2), \\
\overline{F^2(z_\alpha^2 | \lambda_1, \lambda_2)} &= \theta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \left(\sum_\alpha z_\alpha^2 s_2 | e_2, \tau \right) \prod_{\alpha < \beta} [\theta_3(z_\alpha^2 - z_\beta^2 | \tau)]^{\frac{1}{p_1} + p_2}, \quad (59)
\end{aligned}$$

where

$$\begin{aligned}
e_2 &= [p_1(p_1 p_2 + 1)]^{1/2}, s_2 = [p_2 + \frac{1}{p_1}]^{1/2} \\
a_2 &= a_2^* e_2, b_2 = b_2^*, \\
a_2^* &= a_0 + \lambda_1(p_1 p_2 + 1) + \lambda_2 p_1, b_2^* = b_0 \\
\lambda_1 &= 1, \dots, p_1, \lambda_2 = 1, \dots, p_1 p_2 + 1, \\
p_1 N_1 + N_2 &= \phi, N_1 - p_2 N_2 = 0. \quad (60)
\end{aligned}$$

Thus the wave function is

$$\Psi(z_i^1) = \int \prod_{\alpha=1}^{N_2} [dz_\alpha^2] \exp \left(-\frac{\pi\phi (\sum_i [y_i^1]^2 + \frac{2}{p_1} \sum_\alpha [y_\alpha^2]^2)}{\tau_2} \right)$$

$$\begin{aligned}
& \times \prod_{i < j}^{N_1} [\theta_3(z_i^1 - z_j^1 | \tau)]^{p_1} \prod_{\alpha < \beta}^{N_2} \left[\overline{\theta_3(z_\alpha^2 - z_\beta^2 | \tau)} \right]^{p_2} \\
& \times \prod_{i, \alpha}^{N_1, N_2} [\theta_3(z_i^1 - z_\alpha^2 | \tau)] \prod_{\alpha < \beta}^{N_2} |\theta_3(z_\alpha^2 - z_\beta^2 | \tau)|^{\frac{2}{p_1}} \\
& \times \sum_{\lambda_1} \Theta^1(z_i^1, z_\alpha^2 | \lambda_1) \Theta^2(z_i^2 | \lambda_1, \lambda_2), \tag{61}
\end{aligned}$$

where

$$\Theta^1(z_i^1, z_\alpha^2 | \lambda_1) = \theta \left[\begin{matrix} a_1(\lambda_1) \\ b_1 \end{matrix} \right] (\sum_i z_i^1 e_1 + \sum_\alpha z_\alpha^2 e_1^* | e_1, \tau).$$

with all parameters are given by eq. (48), except that $a_1^* = a_0 + \lambda_1$, and

$$\Theta^2(z_i^2 | \lambda_1, \lambda_2) = \overline{\theta \left[\begin{matrix} a_2(\lambda_1, \lambda_2) \\ b_2 \end{matrix} \right] (\sum_\alpha z_\alpha^2 s_2 | e_2, \tau)}$$

where $a_2(\lambda_1, \lambda_2)$ is given by eq. (60). We note that

$$\sum_{\lambda_1} \Theta^1(z_i^1, z_\alpha^2 | \lambda_1) \Theta^2(z_i^2 | \lambda_1, \lambda_2)$$

in eq. (61) is not equal to a theta function on a two dimensional lattice as in the case of $\nu = \frac{1}{p_1 - 1/p_2}$, because $\Theta^2(z_i^2 | \lambda_1, \lambda_2)$ is now an anti-holomorphic function.

As in the case of the corresponding wave function on a sphere, the wave function of eq. (61) also can not be integrated out analytically. Similarly as discussed in the previous section, we can approximate eq. (61) by a wave function which is analytically integrable. We can replace

$$\prod_{\alpha < \beta}^{N_2} |\theta_3(z_\alpha^2 - z_\beta^2 | \tau)|^{\frac{2}{p_1}}$$

in eq. (61) by

$$\exp \left[-\frac{\pi}{2p_1\tau_2} \sum_{\alpha < \beta}^{N_2} (z_\alpha^2 - z_\beta^2 - \bar{z}_\alpha^2 + \bar{z}_\beta^2)^2 \right].$$

4 Conclusions

We have discussed various hierarchical wave functions on a sphere and on a torus. The wave functions can be simplified by using the analytical properties of the wave functions. We also gave a derivation of the hierarchical wave function due to the condensations of QEs on a torus proposed in ref. [10, 11]. The wave function for quasiparticles must be multicomponent even in the case of QEs. We also solved an open problem, the construction of the Laughlin wave function with QE excitations on a torus.

5 Acknowledgments

We would like to thank ICTP for financial supports. We also thank ISI Foundation of Torino for giving us the opportunity to present partial results of this paper in the workshop of “Quantum Hall Effect” (May 25, 1997 to June 14, 1997) at Villa Gualino, Torino.

A Landau Levels on a torus

Consider a magnetic field with potential $\mathbf{A} = -By\hat{x}$. The Hamiltonian is

$$H = \frac{1}{2m}[(p_x + By)^2 + (p_y)^2]. \quad (62)$$

On a torus, we identify $z \sim z + m + n\tau$ with $\tau = \tau_1 + i\tau_2$ and $\tau_2 \geq 0$. The identification will impose boundary conditions on the wave function

$$e^{it_x}\psi = e^{i\phi_1}\psi, e^{i\tau_1 t_x + i\tau_2 t_y}\psi = e^{i\phi_2}\psi, \quad (63)$$

with

$$t_x = p_x, t_y = p_y + Bx \quad (64)$$

as magnetic translation operators and they commute with Hamiltonian. The Dirac quantization condition $\tau_2 B = 2\pi\phi$, with the magnetic flux ϕ being an integer can be derived by requiring operators $e^{it_x}, e^{i\tau_1 t_x + i\tau_2 t_y}$ commuting with each other for the consistence of the boundary conditions of eq. (63).

The wave function describing an electron in the lowest Landau level has the form

$$\psi_l(x, y) = e^{-\frac{By^2}{2}} f(z), \quad (65)$$

where $f(z)$ is the holomorphic function. Higher Landau levels can be obtained by acting operator a^+ on ψ_l , $(a^+)^k \psi_l$, with $a^+ = \partial_z + \frac{B(z-\bar{z})}{4}$. We can write any Landau level state as

$$\psi(x, y) = e^{-\frac{By^2}{2}} f(z, \bar{z}). \quad (66)$$

Now function f is not necessary to be a holomorphic function unless the wave function describes a lowest Landau level states.

By using the relation

$$e^{i\tau_1 t_x + i\tau_2 t_y} = e^{-\frac{i\tau_2 Bx^2}{2\tau_1}} e^{i\tau_1 p_x + i\tau_2 p_y} e^{\frac{i\tau_2 Bx^2}{2\tau_1}}, \quad (67)$$

Eq. (63) can be written as

$$f(z+1, \bar{z}+1) = e^{i\phi_1} f(z, \bar{z}), f(z+\tau, \bar{z}+\bar{\tau}) = e^{i\phi_2} e^{-i\pi\phi(2z+\tau)} f(z, \bar{z}). \quad (68)$$

In the case of lowest landau levels, f is a holomorphic function, and the solutions of eq. (68) are theta functions. For the many-particle wave functions, the condition of eq. (68) shall be imposed on every particle.

B Theta functions

θ function is defined as

$$\theta(z|\tau) = \sum_n \exp(\pi i n^2 \tau + 2\pi i n z), n \in \text{integer}. \quad (69)$$

We will generalize the θ function of eq. (69) to the theta function on the lattice:

$$\theta(z|e, \tau) = \sum_{n_i} \exp(\pi i v^2 \tau + 2\pi i v \cdot z), \quad (70)$$

where v is a vector on a l -dimension lattice (we will call it Λ), $v = \sum_{i=1}^l n_i e_i$, with n_i being integers, $e_i \cdot e_j = A_{ij}$ and z is a vector on the lattice. A_{ij} needs to be a positive definite matrix in order to have a well defined theta function. The θ function in Eq. (69) is a special case of the θ function defined by Eq. (70) with $l = 1, e_1 \cdot e_1 = 1$. We define also

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|e, \tau) = \sum_{n_i} \exp(\pi i (v+a)^2 \tau + 2\pi i (v+a) \cdot (z+b)), \quad (71)$$

where a, b are arbitrary constant vectors on the lattice. The dual lattice e_i^* is defined as (we will call the dual lattice as Λ^*)

$$e_i^* \cdot e_j = \delta_{ij}, \quad (72)$$

then we have $e_i^* \cdot e_j^* = A_{i,j}^{-1}$. One can check that the following relations hold:

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z + e_i | e, \tau) &= e^{2\pi i a \cdot e_i} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z | e, \tau), \\ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z + \tau e_i | e, \tau) &= \exp[-\pi i \tau e_i^2 - 2\pi i e_i \cdot (z + b)] \theta \begin{bmatrix} a \\ b \end{bmatrix} (z | e, \tau), \\ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z + e_i^* | e, \tau) &= e^{2\pi i a \cdot e_i^*} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z | e, \tau), \\ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z + \tau e_i^* | e, \tau) &= \exp[-\pi i \tau (e_i^*)^2 - 2\pi i e_i^* \cdot (z + b)] \theta \begin{bmatrix} a + e_i^* \\ b \end{bmatrix} (z | e, \tau), \end{aligned} \quad (73)$$

and

$$\theta \begin{bmatrix} a + e_i \\ b + e_j^* \end{bmatrix} (z | e, \tau) = \exp(2\pi i a \cdot e_j^*) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z | e, \tau). \quad (74)$$

In a *1-dimension* lattice with $e_1 \cdot e_1 = 1$, $a = b = 1/2$, the θ function is denoted as

$$\theta_3(z|\tau) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z|\tau), \quad (75)$$

is an odd function of z . And we have equations

$$\begin{aligned} \theta_3(z + 1|\tau) &= e^{\pi i} \theta_3(z|\tau), \\ \theta_3(z + \tau|\tau) &= \exp[-\pi i \tau - 2\pi i \cdot (z + \frac{1}{2})] \theta_3(z|\tau). \end{aligned} \quad (76)$$

C Laughlin wave function on a torus

Laughlin wave function on a torus was obtained in ref. [3]. The wave function could be written in a more compact form [13, 10]. we will follow those constructions in refs. [13, 10].

The Laughlin-Jastrow wave function on the torus at the filling $\frac{1}{m}$ (m is an odd positive integer) can be written as

$$\begin{aligned}\Psi(z_i) &= \exp\left(-\frac{\pi\phi\sum_i y_i^2}{\tau_2}\right)F(z_i), \\ F(z_i) &= \theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\left(\sum_i z_i e|e, \tau\right) \prod_{i < j} [\theta_3(z_i - z_j|\tau)]^m, \end{aligned} \quad (77)$$

where θ function is on a 1 -dimension lattice, $e^2 = m$, $i = 1, 2, \dots, N$ with N being the number of the electrons and $a = a^*e^*$, $b = b^*e^*$. Thus

$$\begin{aligned}F(z_i + 1) &= (-1)^{N-1} e^{2\pi a^*} F(z_i), \\ F(z_i + \tau) &= \exp(-\pi(N-1) - 2\pi i b^*) \exp[-i\pi m N(2z_i + \tau)] F(z_i) \end{aligned} \quad (78)$$

Compared to Eq. (68), we get

$$\Phi = mN, \phi_1 = \pi(\phi + 1) + 2\pi n_1 + 2\pi a^*, \phi_2 = \pi(\phi + 1) + 2\pi n_2 - 2\pi b^*. \quad (79)$$

Eq. (79) has solutions

$$\begin{aligned}a^* &= a_0 + i, \quad b^* = b_0, \quad i = 0, 1, \dots, m-1, \\ a_0 &= \frac{\phi_1}{2\pi} + \frac{\phi + 1}{2}, \quad b_0 = -\frac{\phi_2}{2\pi} + \frac{\phi + 1}{2}, \end{aligned} \quad (80)$$

which will give m orthogonal Laughlin-Jastrow wave function (other solutions are not independent on the solutions given by Eq. (80), which can be seen from Eq. (74)). So there is a m -fold center-mass degeneracy.

References

- [1] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.
- [2] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605
- [3] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. B 31 (1985) 2529.
- [4] B. Halperin, Phys. Rev. Lett. 52 (1984) 1583; 52 (1984) 2390(E).
- [5] J.K. Jain, Phys. Rev. Lett. 63 (1989) 199; Phys. Rev. B 41 (1990) 7653; Adv. Phys. 41 (1992) 105.

- [6] Prange and S. Girvin, “The Quantum Hall Effect ”, Springer-Verlag, New York, Heidelberg, 1990, 2nd ed.; and references therein.
- [7] R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin and D. Mahalu, *Nature* 389 (1997) 162; L. Saminadayar, D.C. Glatli, Y. Jin and B. Etienne, *Phys. Rev. Lett.* 79, (1997) 2562.
- [8] N. Read, *Phys. Rev. Lett.* 65 (1990) 1502.
- [9] B. Blok and X.G. Wen, *Phys. Rev. B* 42 (1990) 8133; 42 (1990) 8145; 43 (1991) 8337.
- [10] D. Li, *Int. J. Mod. Phys. B* 7 (1993) 2655.
- [11] E. Keski-Vakkuri and X.G. Wen, *Int. J. Mod. Phys. B* 7 (1993) 4277.
- [12] C. L. de Souza Batista and D. Li, *Phys. Rev. B* 55 (1997) 1582.
- [13] D. Li, *Int. J. Mod. Phys. B* 7 (1993) 2779.
- [14] T. Einarsson, *Phys. Rev. Lett.* 64 (1990) 1995.
- [15] D. Li, *Mod. Phys. Lett. B* 7 (1993) 1103.
- [16] M. Greiter, *Phys. Lett. B* 48 (1994) 3336.
- [17] R. Tao and Y.-S. Wu, *Phys. Rev. B* 30 (1984) 1097; R. Tao and F.D.M. Haldane, *Phys. Rev. B* 33 (1986) 3844; F.D.M. Haldane, *Phys. Rev. Lett.* 55 (1985) 2095.
- [18] D. Mumford, M. Nori and P. Norman, “Tata Lectures on Theta III”, Birkhäuser, Boston 1991.
- [19] R. Iengo and D. Li, *Nucl. Phys. B* **413** (1994) 735 (FS).
- [20] R. Iengo and K. Lechner, *Phys. Rep.* C213 (1992) 179; and references therein.